university of groningen

Exam Computer Assisted Problem Solving (CAPS)

June 28th 2018 9.00-12.00

This exam is 'closed book'. It is NOT allowed to use a book or lecture notes. Only the use of a simple pocket calculator is allowed. Programmable calculators are not allowed, nor the use of electronic devices (tablet, laptop, etc.).

Always give a clear explanation of your answer. An answer without any computation will not be rewarded, so also copy the computations from your scratch paper.

Write your name and student number on each page!

Free points: 10

1. Consider the equation $e^{4x} - 3x^2 = 10$, with solution $x \approx 0.6$.

   (a) 7 (1) Compute 1 iteration with the Secant method, using initial values $x_0, x_1 = 0.5, 1.0$. Determine the most accurate error estimate for x_2.
   (2) Show that the Bisection method, with initial search interval [0.5, 1], reaches a higher accuracy in only one iteration. Explain why Secant is still to be preferred.
   (3) Give an advantage and a disadvantage of Secant, compared to Newton's method.

   (b) 7 Someone uses the iterative method $x_{n+1} = \frac{1}{4}\ln(10 + 3x_n^2)$, with $x_0 = 0.5$. The first 4 iterations are given by

x_0	x_1	x_2	x_3	x_4
0.50000000	0.59372644	0.60077798	0.60134865	0.60139507

   (1) Explain that this method will eventually converge. With which order and factor?
   (2) Determine an error estimate for x_4.
   (3) Calculate an improved solution for x_4 by means of Steffensen extrapolation.

   (c) 4 Design a fast linear method, by introducing a parameter α to the original problem $e^{4x} - 3x^2 = 10$, and computing the optimal value of α

   (d) 7 Give a complete program (in pseudo code or Matlab-like language) that solves the problem using the Newton method, with an accuracy of tol=1E-6. Use an appropriate stopping criterion and start-up procedure.

2. The logarithmic integral, i.e. $Li(x) = \int_2^x \frac{1}{\ln(t)}\,dt$, is used to define $I = Li(e^2) - Li(e)$.

   (a) 9 (1) Write I as one integral and approximate I with the Trapezoidal method on a grid with only 1 segment. Use the global error theorem to give an error estimate. Hint: you may use that f'' has its extreme values at one of the ends.
   (2) Approximate I with Simpson's method on a grid with only 1 segment. Is the error ($\sim h^4$) larger than with Trapezoidal on this grid ($h > 1$)? Explain.

   (b) 8 The integral $I = Li(e^2) - Li(e)$ is approximated with the Midpoint method:

n	$I(n)$
8	3.05446456
16	3.05793644
32	3.05882039
64	3.05904243
128	3.05909801

   $I(n)$ is the approximation of the integral on a grid with n sub-intervals.

P.T.O.

(1) Compute the q-factor. What can you conclude?
(2) Give an error estimate for $I(128)$ based on $I(n)$ values.
(3) Compute improved solutions (T_2) for $I(16)$ and $I(32)$ by means of extrapolation.
(4) Give an error estimate for $T_2(32)$ and compare with the accuracy at (2).
How many Trapezoidal intervals are required (powers of 2) for the same accuracy?
(5) How many function evaluations are needed for $T_2(16)$ (most efficient program)?

(c) [8] Give a complete program (in pseudo code or Matlab-like language) that solves the problem with accuracy tol=1E-6, using Simpson's method (no extrapolation).
Use an appropriate error estimate for the stopping criterion.
Your program should be as computationally efficient as possible.

3. Consider the differential equation $y'(x) = y^2(x) - \frac{1}{2}x$, with boundary condition $y(0) = 1$.

(a) [6] (1) Use Heun's method (RK2) to compute $y(x)$ at x=0.5 on a grid with Δx=0.5.
(2) Which problems arise in case of Implicit(!) Euler on a grid with Δx=0.5?
Does this also occur when the Trapezoidal (Crank-Nicolson) method is used?

(b) [8] With an explicit 2nd order method the solution is determined on 3 grids with $N = 32, 64, 128$ segments. The result at a selection of x locations is as follows

x_n	$N = 32$	$N = 64$	$N = 128$
0.0	1.00000000	1.00000000	1.00000000
0.25	1.31376417	1.31391441	1.31395291
0.5	1.88536590	1.88636234	1.88661946
0.75	3.39836847	3.40770982	3.41016974
1.0	18.80427323	20.52616796	21.16133630

(1) Compute the q-factor for $x = 0.75$. What can you conclude?
(2) Give error estimates for the solutions at $x = 0.75$ and $x = 1.0$ on the finest grid.
Why is the value for $x = 1.0$ larger?
(3) Which N is required (roughly) for an error of 10^{-8} on [0,1]? Really that many?
(4) Compute an improvement for the solution at $x = 0.75$ by means of extrapolation.

(c) [4] When the Midpoint method is evaluated over 2 segments, a so-called 2-step explicit method can be derived. Derive this method (general formulas) and use it to compute the solution of the o.d.e. at $x = 0.5$ (use $y(0)$=1, $y(0.25)$=1.25, gridsize Δx=0.25).

(d) [8] Give a complete program (in pseudo code or Matlab-like language) that solves the problem with accuracy tol=1E-6, using the Heun method (without extrapolation).
Use an appropriate error estimate for the stopping criterion.

4. The bending $w(x)$ (in mm) of a bridge (length L=6 m), due to an uniform load, follows from

$$w''(x) = \alpha\, w(x) + \beta(x - L)x, \quad 0 < x < L$$

with α and β material contants. The bridge is clamped at both ends: $w(0) = w(L) = 0$.

(a) [6] (1) Give the matrix-vector system, when the problem is solved on a grid with $N = 2$ segments by means of the matrix method, using the [1 -2 1]-formula for $w''(x)$.
(2) Take values $\alpha = 0.55$, $\beta = 7.0\text{E-}3$ and solve the system.

(b) [8] Because of partial damage, the left boundary condition changes: $w'(0) = 0$.
(1) Give the new matrix-vector system ($N = 2$), use the same α,β values.
(2) Determine the LU decomposition, which has ones on the main diagonal of U.
(3) Explain why $w'(0) = 0$ requires a finer grid than the situation with $w(0) = 0$.

Total: [100]